Calculus I SI Worksheet

Exam 2 Review

1. $\frac{d}{d\theta}[\sin(\theta)\log_4 \theta^2]$

$$\boxed{\frac{dy}{d\theta} = \cos\theta \cdot \log_4 \theta^2 + \frac{2\sin\theta}{\theta \ln 4}}$$

$\sin\theta$	$\log_4 \theta^2$
$\cos\theta$	$\frac{1}{\theta^2 \ln 4} \cdot \frac{d}{d\theta}(\theta)^2 = \frac{2\theta}{\theta^{2}\ln 4} = \frac{2}{\theta \ln 4}$

2. $\frac{d}{dx}[\operatorname{sech}(\ln(5) - x^2)]$

$$\frac{dy}{dx} = -\operatorname{sech}(\ln 5 - x^2)\tanh(\ln 5 - x^2) \cdot \frac{d}{dx}\overbrace{[\ln 5 - x^2]}^{-2x}$$

$$\boxed{\frac{dy}{dx} = 2x\operatorname{sech}(\ln 5 - x^2)\tanh(\ln 5 - x^2)}$$

3. $\frac{d}{dx}[\frac{x^x \sin(x)}{e^x}]$

$$y = \frac{x^x \sin(x)}{e^x} \qquad \ln y = \ln\left(\frac{x^x \sin(x)}{e^x}\right)$$

$* \ \ln\left(\frac{a}{b}\right) = \ln a - \ln b$

$\ln(a \cdot b) = \ln a + \ln b$

$$\ln y = \ln(x^x \sin x) - \cancel{\ln} e^x \xrightarrow{} x = \ln x^x + \ln \sin x - x = x \ln x + \ln \sin x - x$$

$$\frac{d}{dx}\left[\ln y = x \ln x + \ln \sin x - x\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \ln x + 1 + \frac{1}{\sin x}\cos x - 1$$

$\frac{1}{\sin x}\cos x \to \cot x$

$$\frac{dy}{dx} = [\ln x \cancel{+1} + \cot x \cancel{-1}]\, y$$

$* \ y = \frac{x^x \sin x}{e^x}$

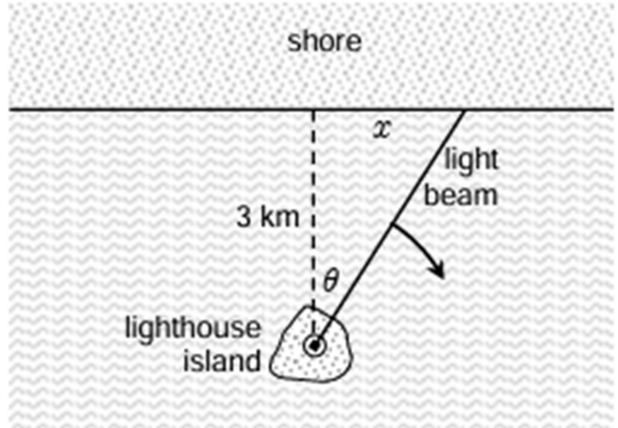
$$\boxed{\frac{dy}{dx} = [\ln x + \cot x]\frac{x^x \sin x}{e^x}}$$

4. $\frac{d}{dy}[\frac{\mathrm{csch}(y)+y}{e^y-2}]$

$\mathrm{csch}\, y + y$	$e^y - 2$
$-\mathrm{csch}\, y \coth y + 1$	e^y

$$= \boxed{\frac{(e^y-2)(-\mathrm{csch}\, y \coth y + 1) - (\mathrm{csch}\, y + y)e^y}{(e^y-2)^2}}$$

5. A lighthouse sits on a small island near a rocky shoreline, emitting a rotating beam of light. The lighthouse is 3 km from the shore, and it emits a beam of light that rotates at a rate of 8π rad/min.



(a) Find a formula for x as a function of θ.

(b) Take the derivative of your formula from part (a) to find a formula for $\frac{dx}{dt}$ in terms of θ and $\frac{d\theta}{dt}$.

(c) How quickly is the end of the light beam moving along the shoreline when θ = π/6 rad?

$$\tan\theta = \frac{O}{A}$$

$$\tan\theta = \frac{x}{3}$$

a) $\boxed{x = 3\tan\theta}$

$$\frac{d}{dt}\left[x = 3\tan\theta\right]$$

b) $\boxed{\frac{dx}{dt} = 3\sec^2\theta \, \frac{d\theta}{dt}}$

$$\frac{dx}{dt} = 3\sec^2\left[\frac{\pi}{6}\,\text{rad}\right] \cdot \frac{8\pi\,\text{rad}}{\text{min}}$$

c) $\boxed{\frac{dx}{dt} = 32\pi\,\frac{\text{km}}{\text{min}} \approx 100.53\,\frac{\text{km}}{\text{min}}}$

6. In physics, the energy stored in a stretched spring is determined by the equation

$$E = \frac{1}{2}kx^2$$

where E is the energy, k is a constant (the "spring constant"), and x represents the distance that the spring has been stretched.

(a) Find a formula for $\frac{dE}{dt}$ in terms of k, x, and $\frac{dx}{dt}$.

(b) A spring with spring constant k = 0.2 Joules/cm2 is being stretched at a rate of 1.5 cm/sec. How quickly is the energy stored in the spring increasing at the moment that x = 10 cm?

$$\frac{d}{dt}\left[E = \frac{1}{2}kx^2\right]$$

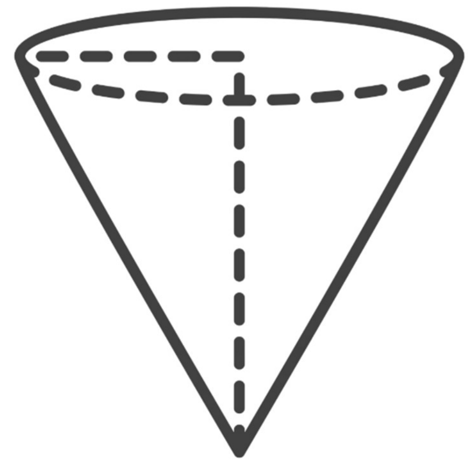
a) $\boxed{\frac{dE}{dt} = kx\frac{dx}{dt}}$

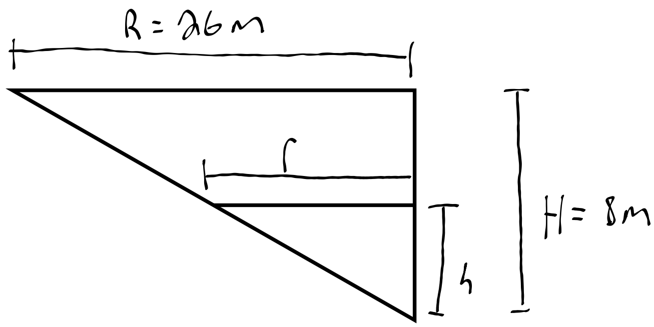
$$\frac{dE}{dt} = \left(0.2\ \frac{\text{Joules}}{\cancel{\text{cm}}^2}\right)\left(10\ \cancel{\text{cm}}\right)\left(1.5\ \frac{\cancel{\text{cm}}}{s}\right)$$

b) $\boxed{\frac{dE}{dt} = 3\ \frac{J}{s} = 3W}$

$*\left[\frac{J}{s}\right] = [W]$

7. A tank of water in the shape of a cone is being filled with water at a rate of 12 m^3/sec. The base radius of the tank is 26 meters, and the height of the tank is 8 meters. At what rate is the depth of the water in the tank changing when the radius of the top of the water is 10 meters?





$$\frac{r}{h} = \frac{R}{H} = \frac{26}{8}$$

$$r = \frac{26}{8}h = 3.25h$$

$$V_{cone} = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi (3.25h)^2 h$$

$$\frac{d}{dt}\left[V = \frac{3.25^2}{3}\pi h^3\right]$$

@ $r = 10m$

$$h = \frac{r}{3.25} = \frac{10}{3.25} = 3.08m$$

$$\frac{dV}{dt} = 3.25^2\pi h^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{\frac{dV}{dt}}{3.25^2\pi h^2} = \frac{12\frac{m^3}{s}}{3.25^2\pi(3.08m)^2}$$

$$\boxed{\frac{dh}{dt} = 0.038\frac{m}{s}}$$

8. Prove that the derivative of tanh(x) is sech^2(x) given that sinh(x) = $\frac{e^x - e^{-x}}{2}$ & cosh(x) = $\frac{e^x + e^{-x}}{2}$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{\frac{e^x - e^{-x}}{\cancel{2}}}{\frac{e^x + e^{-x}}{\cancel{2}}} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\frac{d}{dx}\left[\frac{e^x - e^{-x}}{e^x + e^{-x}}\right] \qquad \frac{e^x - e^{-x}}{e^x + e^{-x}} \,\Big|\, \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$= \frac{[e^x + e^{-x}]^2 - [e^x - e^{-x}]^2}{[e^x + e^{-x}]^2}$$

$$= \cancelto{1}{\frac{[e^x + e^{-x}]^2}{[e^x + e^{-x}]^2}} - \frac{[e^x - e^{-x}]^2}{[e^x + e^{-x}]^2} = 1 - \overbrace{\left[\frac{e^x - e^{-x}}{e^x + e^{-x}}\right]}^{\tanh x}{}^2$$

hyperbolic pythagorean identity

$$= 1 - \tanh^2 x \qquad * \; 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\boxed{\frac{d}{dx}[\tanh x] = \operatorname{sech}^2 x}$$